Družicové metody v geodézii a katastru
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GRAIL and LOLA satellite data resolve the long-lasting convergence/divergence problem for the analytical downward continuation of the external spherical harmonic expansions

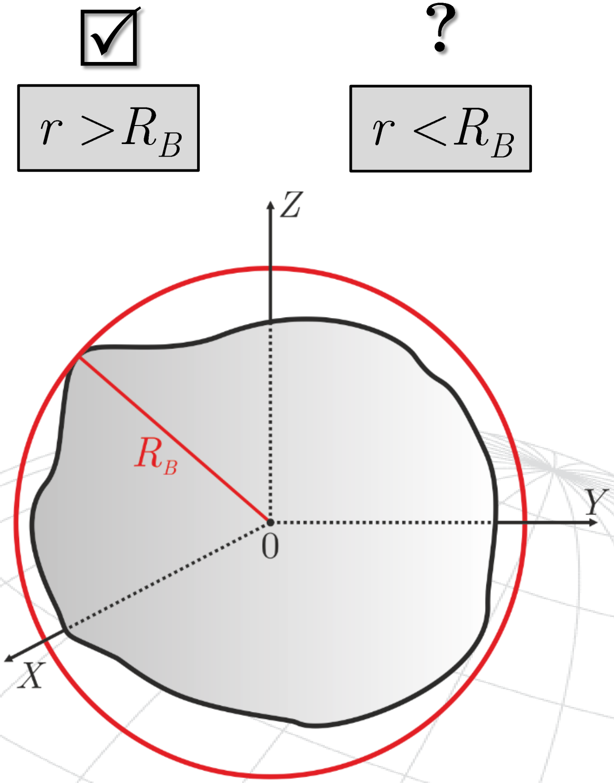
Michal Šprlák(1), Shin-Chan Han(2), Martin Pitoňák(1), Pavel Novák(1)

(1) – NTIS – *New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Plzeň, Czech Republic*

(2) – *School of Engineering, Faculty of Engineering and Built Environment, University of Newcastle, Callaghan, NSW, Australia*

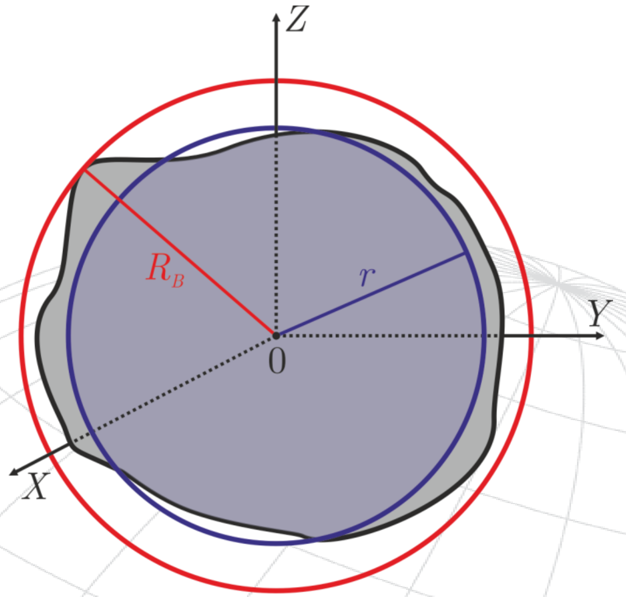
1. Introduction:

- External Spherical Harmonic Expansions (SHEs) – popular mathematical apparatus,
- Converge $\forall r > R_B$, but employed @ $r < R_B$,
- Justification of the analytical continuation questionable,
- Contradictory conclusions by theoreticians and pragmatists.



2. This presentation:

- Lunar gravitational field inferred by topographic masses of constant density,
- Assumption: perfect correlation with observations, e.g., from GRAIL,
- Investigation of two gravitational field quantities in spectral and spatial domain,
- Analysis @ $r = 1738$ km $< R_B = 1748.2$ km.



3. First-order radial derivative of V:

External SHE:

$$V_r(r,\Omega) = -\frac{GM}{R^2}\sum_{n,m}^{N}\left(\frac{R}{r}\right)^{n+2}(n+1)\,\bar{C}_{n,m}\,\bar{Y}_{n,m}(\Omega)$$

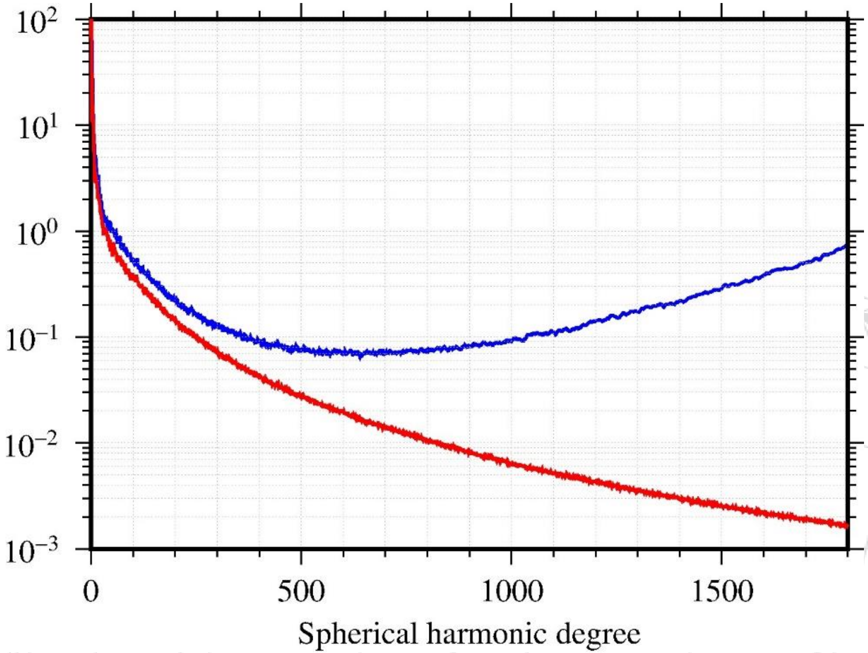
Internal SHE:

$$V_r(r,\Omega) = -\frac{GM}{R_o^2}\sum_{n,m}^{N}\left(\frac{R_o}{r}\right)^{n+2}(n+1)\,\bar{C}^o_{n,m}(r)\,\bar{Y}_{n,m}(\Omega)$$

$$+\frac{GM}{R_i^2}\sum_{n,m}^{N}\left(\frac{r}{R_i}\right)^{n-1} n\,\bar{C}^i_{n,m}(r)\,\bar{Y}_{n,m}(\Omega)$$

r, Ω - spherical geocentric coordinates,
GM - planetocentric gravitational constant,
$\bar{Y}_{n,m}$ - spherical harmonic function of degree n and order m,
$\bar{C}_{n,m}$, $\bar{C}^o_{n,m}$, $\bar{C}^i_{n,m}$ - spherical harmonic coefficients of degree n and order m,
R, R_o, R_i - scale factors,
N - maximum degree of the expansion.

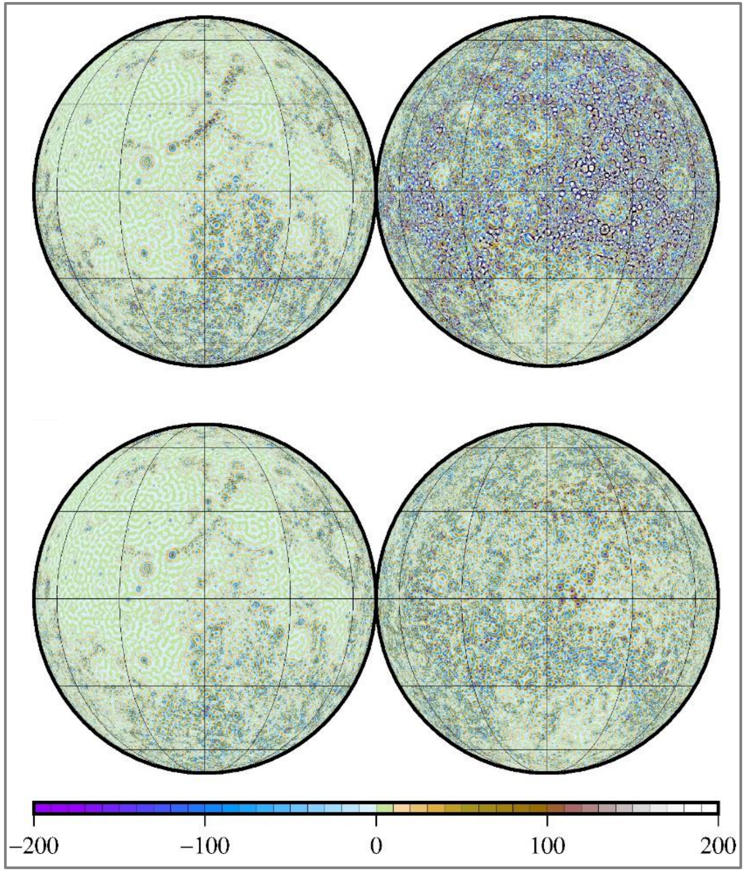
Square-root of degree-order variances [mGal]:



Nearside Farside

V_r

V_r



Statistics (in mGal)

Model	Min.	Max.	Mean	Std. dev.
V_r	-1354.94	1333.11	-0.16	64.73
V_r	-248.83	274.95	-0.18	39.72

(Spherical harmonic spectrum 150-600)

Correlation coefficients

Quantity	Nearside	Farside	Global
V_r wrt. V_r	0.462	-0.135	0.050

4. Second-order radial derivative of V:

External SHE:

$$V_{rr}(r,\Omega) = \frac{GM}{R^3} \sum_{n,m}^{N} \left(\frac{R}{r}\right)^{n+3} (n+1)(n+2)\, \bar{C}_{n,m}\ \bar{Y}_{n,m}(\Omega)$$

Internal SHE:

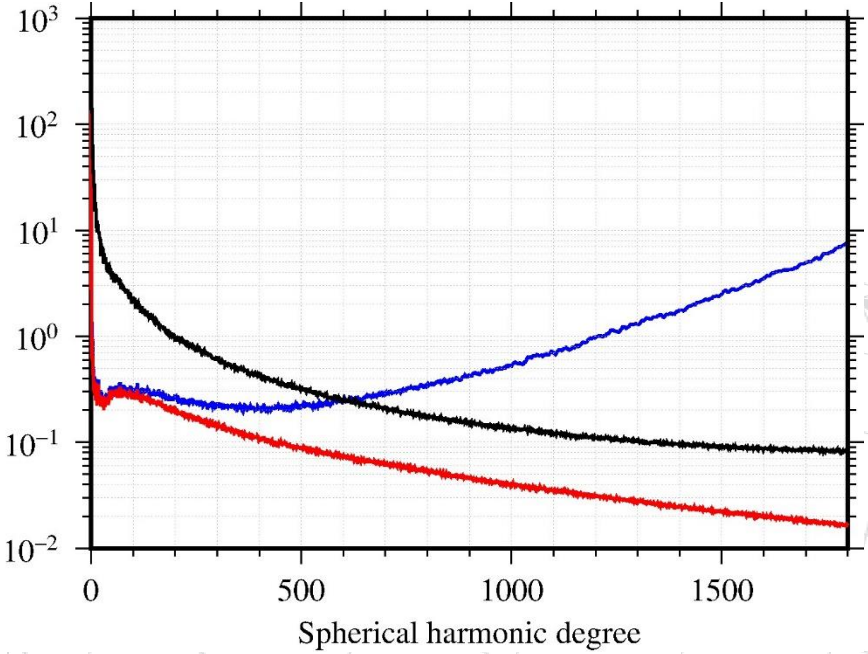
$$V_{rr}(r,\Omega) = \frac{GM}{R_o^3} \sum_{n,m}^{N} \left(\frac{R_o}{r}\right)^{n+3} (n+1)(n+2)\, \bar{C}^{o}_{n,m}(r)\, \bar{Y}_{n,m}(\Omega)$$

$$+ \frac{GM}{R_i^3} \sum_{n,m}^{N} \left(\frac{r}{R_i}\right)^{n-2} n(n-1)\, \bar{C}^{i}_{n,m}(r)\, \bar{Y}_{n,m}(\Omega)$$

$$- \frac{4\pi GM}{R_\varrho^3} \sum_{n,m}^{N} \bar{\varrho}_{n,m}(r)\, \bar{Y}_{n,m}(\Omega)$$

$\bar{\varrho}_{n,m}$ - spherical harmonic coefficients of density,
R_ϱ - scale factor.

Square-root of degree-order variances [E]:

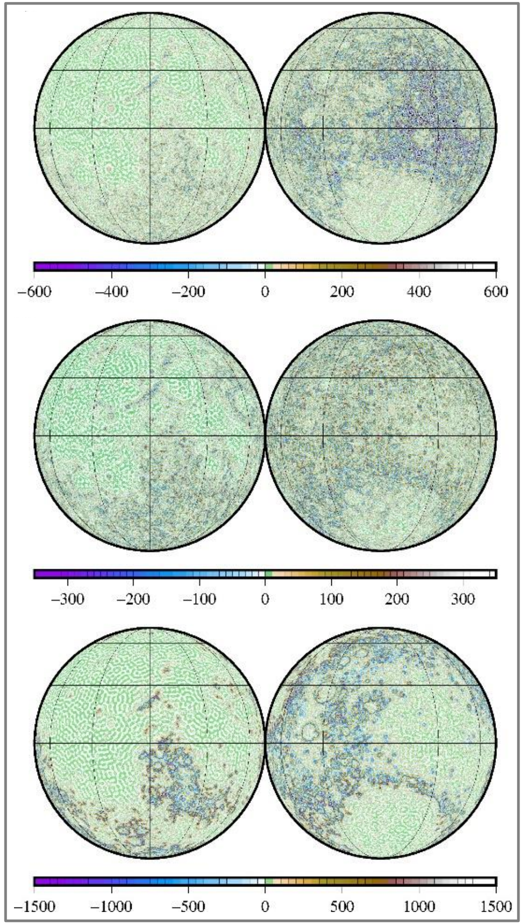


Nearside Farside

V_{rr}

V_{rr}

V_{rr} + DT



Statistics (in E)

Quantity	Min.	Max.	Mean	Std. dev.
V_{rr}	-3753.14	3134.32	0.30	120.69
V_{rr}	-600.69	559.45	0.32	72.77
V_{rr} + DT	-2283.12	2458.96	0.09	306.39

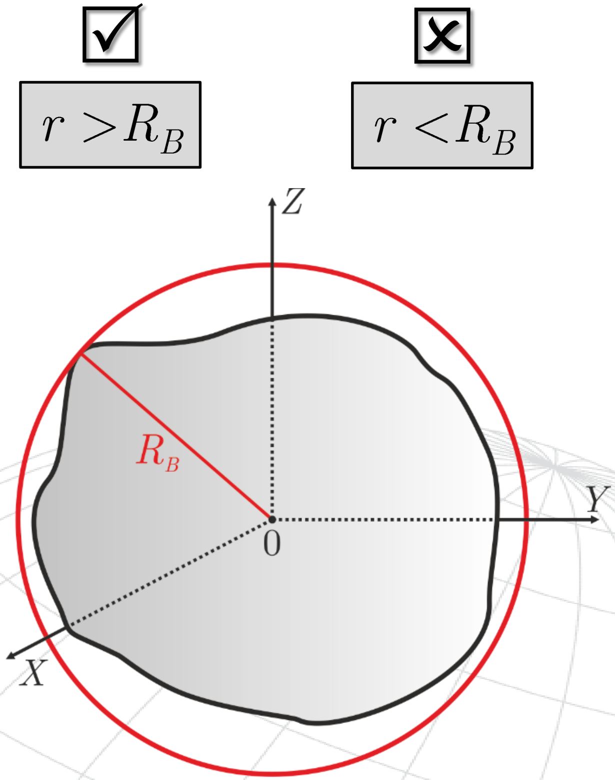
(Spherical harmonic spectrum 150-600)

Correlation coefficients

Quantity	Nearside	Farside	Global
V_{rr} wrt. V_{rr}	0.927	0.737	0.778
V_{rr} + DT wrt. V_{rr}	-0.348	-0.225	-0.266

5. Conclusions:

- Justification of the analytical continuation examined,
- Failure of the external SHEs @ $r < R_B$ demonstrated,
- Density cannot be neglected when calculating V_{rr},
- Implications for future gravitational field modelling and its applications.



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On the use of spherical harmonic series inside the minimum Brillouin sphere: Theoretical review and evaluation by GRAIL and LOLA satellite data

Michal Šprlák[a,*], Shin-Chan Han[b]

[a] NTIS – New Technologies for the Information Society, Faculty of Applied Sciences, University of West Bohemia, Technická 8, 306 14 Plzeň, Czech Republic

[b] School of Engineering, Faculty of Engineering and Built Environment, University of Newcastle, University Drive, Callaghan, NSW 2308, Australia

Thank you for your attention!